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A Singular Eigenfunction Expansion in Anisotropic Transport Theory*

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1. INTRODUCTION

Let K be a compact symmetric integral operator on $L_2(-1, 1)$ with kernel $k(u, u')$ which is real and analytic on $-1 \leqslant u, u' \leqslant 1$. Consider the eigenvalue problem

$$(s - u)\, g_s(u) = sKg_s = s \int_{-1}^{1} k(u, u')\, g_s(u')\, du', \tag{1}$$

where s is a complex number. We shall show that if $f(u)$ is Hölder continuous on $-1 < u < 1$ then f can be expanded in the eigenfunctions of (1) provided that one admits generalized functions (distributions) as singular eigenfunctions corresponding to the continuous spectrum $-1 < s < 1$. This method leads to singular integrals in the expansion theorem.

Equation (1) comes from the one-dimensional one velocity neutron transport equation after a separation of variables. These matters are discussed in [1]. K. M.. Case [2] first proved the expansion theorem when $k(u, u') = \text{const.}$ Rosenbaum [3] has extended Case's results to the case where $k(u, u') = g(u)\, h(u')$.

In Section V we shall indicate briefly how the expansion theorem may be used to reduce boundary value problems in transport theory to systems of singular integral equations. However, a detailed investigation of this method of solving transport problems will be reserved for a later paper.

II. PRELIMINARIES

The transport equation in one dimension is

$$+\, u \frac{\partial f}{\partial x}(x, u) + f(x, u) = \int_{-1}^{1} k(u, u'')\, f(x, u'')\, du'', \tag{2}$$

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where x is the spatial variable; $u = \cos\theta$ is the direction number of the velocity of the neutron (θ is the angle between the velocity of the neutron and the positive x-axis); $f(x, u)$ is the density of neutrons at x with direction number u; $k(u, u')$ is related to the scattering kernel. If we look for solutions to (2) of the form $f(x, u) = e^{-x/s} g_s(u)$ we are led to the integral equation (1).

Let $L_2(-1, 1)$ be the Hilbert space of complex valued functions with the standard (complex) inner product. We shall say that s_0 is an eigenvalue of (1) if there exists a function $\varphi_0(u) \in L_2(-1, 1)$ which satisfies (1) with $s = s_0$. The value $s_0 = \infty$ will be called an eigenvalue if there is a function φ_∞ satisfying $(I - K)\varphi_\infty = 0$, where I is the identity operator.

It will be assumed throughout that the kernel $k(u, u')$ is analytic in both variables in a domain $\mathscr{D}$ in the complex u-plane containing the segment $[-1, 1]$. It was shown in [4] that under this assumption there are only a finite number of eigenvalues of (1) in the extended complex plane. These eigenvalues will be denoted by s_i, $i = 1, \cdots, n$ (with s_i repeated if there is more than one eigenfunction with the eigenvalue $s = s.$) There is also a continuous spectrum $-1 \leqslant s \leqslant 1$ due to the operator $u : f(u) \to uf(u)$. This will be seen after we have derived some distributions which have formal properties typically associated with eigenfunctions of (1). We shall assume that no eigenvalues of (1) are embedded in the continuous spectrum, but we shall allow $s = \infty$ to be an eigenvalue.

We shall need the following results in the theory of singular integral equations.

LEMMA 2.1. *Let $q(u, u')$ be Hölder continuous on $-1 \leqslant u, u' \leqslant 1$. Then*

$$\text{(i)} \qquad \int_{-1}^{1} P \int_{-1}^{1} \frac{q(u, u')}{u - u'}\, du'\, du = \int_{-1}^{1} P \int_{-1}^{1} \frac{q(u, u')}{u - u'}\, du\, du'$$

$$\text{(ii)} \qquad \int_{-1}^{1} P \int_{-1}^{1} \frac{q(u, u')}{t - u'}\, du'\, du = P \int_{-1}^{1} \frac{du'}{t - u'} \int_{-1}^{1} q(u, u')\, du$$

$$\text{(iii)} \quad P \int_{-1}^{1} \frac{du}{t - u} P \int_{-1}^{1} \frac{q(u, s)}{s - u}\, ds = \pi^2 q(t, t) + P \int_{-1}^{1} ds\, P \int_{-1}^{1} \frac{q(u, s)\, du}{(t - u)(s - u)}.$$

LEMMA 2.2. *Let $q(u)$ be Hölder continuous on $-1 \leqslant u \leqslant 1$, and let*

$$Q(s) = P \int_{-1}^{1} \frac{q(u)}{s - u}\, du.$$

Then $Q(s)$ is Hölder continuous on $-1 < s < 1$ and has logarithmic singularities at $s = \pm 1$. If $q(u)$ vanishes at $u = \pm 1$, then $Q(s)$ is bounded at $s = \pm 1$.

Lemmas 2.1 and 2.2 are standard results whose proofs may be found in [5]; Lemma 2.1 (iii) is known as the Poincaré-Bertrand formula.

III. The Expansion Theorem

Let s_i be an eigenvalue of (1). Let the functions φ_{ij}, $j = 1, 2, \cdots$ satisfy the system of equations

$$\begin{aligned} &(u - s_i(I - K))\, \varphi_{ij} = 0, \\ &(u - s_i(I - K))\, \varphi_{ij} = (I - K)\, \varphi_{ij-1} \qquad j = 2, 3, \cdots \end{aligned} \tag{3}$$

if $s_i \neq \infty$; or

$$\begin{aligned} &(I - K)\, \varphi_{i1} = 0 \\ &(I - K)\, \varphi_{ij} = u\varphi_{ij-1} \qquad j = 2, 3, \cdots \end{aligned} \tag{3'}$$

if $s_i = \infty$. The functions φ_{ij} will be called a system of h.o. (higher order) eigenfunctions for the eigenvalue s_i. It will be seen later that the system $\{\varphi_{ij}\}$ is finite; let $j = 1, 2, \cdots, m_i$.

Consider the operator $R = (I - K)^{-1}$ defined on the subspace $\mathscr{R}$ = range of $(I - K)$. The operator R will be defined everywhere if and only if ∞ is not an eigenvalue of (1). In that case, system (3) can be written

$$\begin{aligned} &(s_i - Ru)\, \varphi_{i1} = 0 \\ &(s_i - Ru)\, \varphi_{ij} = \varphi_{ij-1}\, . \end{aligned}$$

If we let $\mathscr{M}_i = [\varphi_{i1}, \varphi_{i2}, \cdots]$ then the operator Ru is defined on $\mathscr{M}_i$, $\mathscr{M}_i$ is invariant under Ru, and the φ_{ij} form a Jordan Canonical basis for Ru restricted to $\mathscr{M}_i$. If $s_i = \infty$, then $\mathscr{M}_i = [\varphi_{i1}, Ru_{i1}, (Ru)^2\, \varphi_{i1}, \cdots]$.

We can also find a linear manifold $\mathscr{M}_0$ invariant under Ru corresponding to the continuous spectrum $-1 \leqslant s \leqslant 1$. Following K. M. Case [2], let us look for distributions of the form

$$\varphi_s(u) = C(s)\, \delta(s - u) + sP\, \frac{\varphi(u, s)}{s - u}$$

which formally satisfy the eigenvalue problem (1) when $-1 < s < 1$. Here $\delta(s - u)$ is the Dirac function and P indicates that we are to take the Cauchy Principal Value of the integral involved. $C(s)$ is a Hölder continuous function to be determined. The distribution $C(s)\, \delta(s - u)$ is a solution of the homogeneous equation $(s - u)\, \varphi_s(u) = 0$. The distribution $sP\, [\varphi(u, s)/(s - u)]$ is a particular solution to the inhomogeneous equation $(s - u)\, \varphi_s = sK\varphi_s$. The procedure here is purely formal; its justification is simply that it leads to distributions which behave like eigenfunctions in some suitable sense.

Let us find distributions φ_s which formally solve (1); we have

$$(s - u)\,\varphi_s(u) = s\varphi(u, s) = s\int_{-1}^{1} k(u, u')\,\varphi_s(u')\,du',$$

$$\varphi(u, s) = k(u, s)\,C(s) + sP\int_{-1}^{1} \frac{k(u, u')\,\varphi(u', s)}{s - u'}\,du'. \tag{4}$$

We must find functions $C(s)$ and $\varphi(u, s)$ which together satisfy (4). Write

$$\varphi(u, s) = k(u, s)\,C(s) + s\int_{-1}^{1} \frac{k(u, u') - k(u, s)}{s - u'}\,\varphi(u', s)\,du'$$
$$+ sk(u, s)\,P\int_{-1}^{1} \frac{\varphi(u', s)}{s - u'}\,du'. \tag{5}$$

Let the kernel appearing in the integral operator in (5) be denoted by

$$h(u, u', s) = s\,\frac{k(u, u') - k(u, s)}{s - u'}. \tag{6}$$

The function h is analytic for $u, u', s \in \mathscr{D}$.

It has been shown [4] that the Fredholm integral equation

$$F(u, s) - \int_{-1}^{1} h(u, u', s)\,F(u', s)\,du' = k(u, s) \tag{7}$$

has a unique solution $F(u, s)$ which is analytic in u and has poles in s, for $u, s \in \mathscr{D}$. The poles of $F(u, s)$ are eigenvalues of the original integral equation. Let us take $\varphi(u, s)$ to be the solution of (7) and set

$$C(s) = 1 - sP\int_{-1}^{1} \frac{\varphi(u', s)}{s - u'}\,du'. \tag{8}$$

This choice of $\varphi(u, s)$ and $C(s)$ provides a solution of (4).

We shall assume that $[C(s)]^2 + \pi^2 s^2 \varphi^2(s, s) \neq 0$ on $-1 < s < 1$. This is equivalent [4] to the assumption that there are no eigenvalues embedded in the continuous spectrum.

From now on, if $F(u)$ is Hölder continuous on $-1 < u < 1$ we shall often write $(F, \varphi_s(u))$ for

$$\int_{-1}^{1} F(u')\,\varphi_s(u')\,du' = C(s)\,F(s) + sP\int_{-1}^{1} \frac{F(u')\,\varphi(u', s)}{s - u'}\,du'.$$

THEOREM 3.1. *Let $F(s)$ be Hölder continuous on $-1 < s < 1$ and let*

$$f(u) = \int_{-1}^{1} F(s)\,\varphi_s(u)\,ds = C(u)\,F(u) + P\int_{-1}^{1} \frac{sF(s)\,\varphi(u, s)}{s - u}\,ds$$

Then f is in the domain of Ru and

(i) $(Ru)f = \int_{-1}^{1} sF(s)\, \varphi_s(u)\, ds.$

If $g(u)$ is any Hölder continuous function on $-1 < u < 1$, then

(ii) $$(ug, \varphi_s) = s((I-K)\, g, \varphi_s).$$

If φ_{ij} are the h.o. eigenfunctions of (1), then

(iii) $$(u\varphi_{ij}, \varphi_s) = 0.$$

PROOF. To prove (i) we make the calculation

$$\begin{aligned}
&(I-K)\int_{-1}^{1} sF(s)\, \varphi_s(u)\, ds \\
&= uC(u)\, F(u) + P\int_{-1}^{1} \frac{s^2F(s)\, \varphi(u,s)}{s-u}\, ds \\
&\quad - \int_{-1}^{1} k(u,u')\, u'C(u')\, F(u')\, du' \\
&\quad - \int_{-1}^{1} k(u,u')\, P\int_{-1}^{1} \frac{s^2F(s)\, \varphi(u',s)}{s-u'}\, ds\, du'.
\end{aligned}$$

Change the dummy variable of integration from u' to s in the third term and invert the order of integration in the fourth term (Lemma 2.1 (i)). Then we get, using the integral equation (4) for $\varphi(u, s)$ in the last two terms,

$$\begin{aligned}
uC(u)\, F(u) &+ P\int_{-1}^{1} \frac{s^2F(s)\, \varphi(u,s)}{s-u}\, ds - \int_{-1}^{1} sF(s)\, \varphi(u,s)\, ds \\
&= uC(u)\, F(u) + uP\int_{-1}^{1} \frac{sF(s)\, \varphi(u,s)}{s-u}\, ds = uf(u).
\end{aligned}$$

It now follows that $uf \in \mathscr{R}$ and hence that (i) holds.

To prove (ii) we have formally

$$(ug, \varphi_s) = (g, u\varphi_s) = (g, s(I-K)\, \varphi_s) = s((I-K)\, g, \varphi_s).$$

This procedure can be justified rigorously in much the same way that part (i) was proved. Again the integral equation (4) for $\varphi(u, s)$ will have to be used; but this time Lemma 2.1 (ii) must be applied.

If $s_i \neq \infty$, we have from (ii) above that

$$(u\varphi_{i1}, \varphi_s) = s((I-K)\, \varphi_{i1}, \varphi_s) = \frac{s}{s_i}(u\varphi_{i1}, \varphi_s),$$

which implies that $(u\varphi_{i1}, \varphi_s) = 0$ for $-1 < s < 1$. The orthogonality of the h.o. eigenfunctions is proved by induction. Suppose that $(u\varphi_{ij-1}, \varphi_s) = 0$, $-1 < s < 1$. Then

$$\begin{aligned} s(u\varphi_{ij}, \varphi_s) &= s(s_i(I-K)\varphi_{ij} + (I-K)\varphi_{ij-1}, \varphi_s) \\ &= (s_i\varphi_{ij} + \varphi_{ij-1}, s(I-K)\varphi_s) \\ &= s_i(u\varphi_{ij}, \varphi_s) + (u\varphi_{ij-1}, \varphi_s) \\ &= s_i(u\varphi_{ij}, \varphi_s). \end{aligned}$$

This shows that $(u\varphi_{ij}, \varphi_s) = 0$ for $-1 < s < 1$.

The proof for eigenfunctions corresponding to $s = \infty$ is similar. If $(I-K)\varphi_{i1} = 0$, for example, then

$$(u\varphi_{i1}, \varphi_s) = s((I-K)\varphi_{i1}, \varphi_s) = 0.$$

The above theorem shows that any eigenfunction of (1) must be a solution of the homogeneous singular integral equation $(uf, \varphi_s) = 0$ for $-1 < s < 1$. The following theorem is the converse to this fact and essentially proves the completeness of the eigenfunctions of (1).

THEOREM 3.2. *Let $\mathcal{N}_0$ be the linear manifold of functions $f(u)$ which are Hölder continuous on $-1 < u < 1$ and which satisfy $(uf, \varphi_s) = 0$ for $-1 < s < 1$. Then $\mathcal{N}_0$ is finite dimensional and consists precisely of the eigenfunctions and h.o. eigenfunctions of (1).*

PROOF. If $f \in \mathcal{N}_0$ then f satisfies the singular integral equation

$$C(s)f(s) + P\int_{-1}^{1} \frac{uf(u)\varphi(u, s)}{s-u}\,du = 0, \qquad -1 < s < 1.$$

Since there are at most a finite number of solutions to this equation [5], it follows that $\mathcal{N}_0$ is finite dimensional; let $N = \dim \mathcal{N}_0$. By Theorem 3.1 (iii) all the φ_{ij} belong to $\mathcal{N}_0$; hence there are only a finite number of h.o. eigenfunctions. We shall now show that $\mathcal{N}_0$ has a basis consisting of h.o. eigenfunctions of (1).

Let $\mathcal{N}_1$ be the linear manifold of Hölder continuous functions satisfying $(f, \varphi_s) = 0$. If $f \in \mathcal{N}_0$, then $uf \in \mathcal{N}_1$. Conversely, if $f \in \mathcal{N}_1$ then it is easily seen that for $s \neq 0$,

$$\frac{f(s)}{s} = -\frac{P\displaystyle\int_{-1}^{1} \frac{f(u)\varphi(u, s)}{s-u}\,du}{C(s)}.$$

Since $C(0) = 1$ the right side is Hölder continuous at $s = 0$. Therefore $f(u)/u \in \mathcal{N}_0$, and $u\,\mathcal{N}_0 = \mathcal{N}_1$. If $\{f_i\}$ is a basis for $\mathcal{N}_0$ then $\{uf_i\}$ is a basis for $\mathcal{N}_1$.

Let $f \in \mathcal{N}_0$. From Theorem 3.1 (ii) we have

$$s((I - K)f, \varphi_s) = (uf, \varphi_s) = 0.$$

If $s \neq 0$ this implies that $((I - K)f, \varphi_s) = 0$; since $((I - K)f, \varphi_s)$ is Hölder continuous in s, $((I - K)f, \varphi_s) = 0$ when $s = 0$ also. It follows that $(I - K)f \in u\mathcal{N}_0 = \mathcal{N}_1$. Let us represent the transformation $(I - K) : \mathcal{N}_0 \rightarrow \mathcal{N}_1$ by an $N \times N$ matrix (A). If we suppose that the basis for $\mathcal{N}_0$ has been chosen so that (A) is in Jordan Canonical form, we shall have

$$(I - K)\begin{vmatrix} f_1 \\ \vdots \\ f_N \end{vmatrix} = \begin{vmatrix} \begin{array}{ccccc} t_1 & 0 & 0 & \cdots & \\ 1 & t_1 & 0 & \cdots & \\ \vdots & & & & \\ & & & & 0 \\ 0 & & \cdots & 1 & t_1 \end{array} & \\ & \ddots \\ & & \begin{array}{ccccc} 0 & 0 & \cdots & & \\ 1 & 0 & \cdots & & \\ \vdots & & & & \\ 0 & & \cdots & 1 & 0 \end{array} \end{vmatrix} \begin{vmatrix} uf_1 \\ \vdots \\ uf_N \end{vmatrix}$$

Corresponding to each block of (A) are functions which satisfy a system of equations of the form

$$\begin{aligned} (I - K)f_{k1} &= t_k u f_{k1} \\ (I - K)f_{kj+1} &= t_k u f_{kj+1} + u f_{kj} \qquad j = 2, 3, \cdots . \end{aligned} \tag{9}$$

Thus the diagonal entries of (A), the t_k, are reciprocals of the eigenvalues of (1); zeroes on the diagonal of (A) correspond to the eigenvalue $s = \infty$.

Let $\mathcal{M}_k' = [f_{k1}, f_{k2}, \cdots]$. New linear combinations of the $\{f_{kj}\}$ (k fixed) can therefore be formed to construct a basis for $\mathcal{M}_k'$ which satisfies the system of equations (3); the members of this new basis form a system of h.o. eigenfunctions, Q.E.D. The reason for preferring the system of equations (3) will be made clear in Section IV.

We shall need one more theorem in preparation for the eigenfunction expansion theorem.

THEOREM 3.3. *Suppose $f(u)$ is Hölder continuous on $-1 < u < 1$ and that*

$$f(u) = \int_{-1}^{1} F(s)\, \varphi_s(u)\, ds, \qquad -1 < u < 1,$$

where $F(s)$ *is Hölder continuous on* $-1 < s < 1$. *Then*

$$t[C^2(t) + \pi^2 t^2 \varphi^2(t, t)] F(t) = (uf, \varphi_t), \qquad -1 < t < 1.$$

PROOF. A straightforward calculation shows that if f is as in the theorem, then

$$\int_{-1}^{1} uf(u)\, \varphi_t(u)\, du$$

$$= tC^2(t) F(t) + tC(t)\, P \int_{-1}^{1} \frac{sF(s)}{s-t} \varphi(t, s)\, ds$$

$$- tP \int_{-1}^{1} \frac{sF(s)}{s-t} \varphi(s, t)\, C(s)\, ds$$

$$+ tP \int_{-1}^{1} \frac{u\varphi(u, t)}{t-u}\, du P \int_{-1}^{1} \frac{sF(s)\, \varphi(u, s)}{s-u}\, ds.$$

Here we have changed the dummy variable of integration from u to s in the third term on the right. Now apply Lemma 3.1 (iii) to the iterated singular integral. After some manipulation we get

$$(uf, \varphi_t) = t[C^2(t) + \pi^2 t^2 \varphi^2(t, t)] F(t) + P \int_{-1}^{1} \frac{stF(s)}{s-t} I(s, t)\, ds,$$

where

$$I(s, t) = C(t)\, \varphi(t, s) - C(s)\, \varphi(s, t) + P \int_{-1}^{1} u\varphi(u, t)\, \varphi(u, s) \left[\frac{1}{t-u} - \frac{1}{s-u}\right] du.$$

Now $I(t, t) = 0$; we shall show that $I(s, t)$ vanishes identically in s on $-1 < s < 1$. Using (8) for $C(s)$ we can show that

$$I(s, t) = \varphi(t, s) - \varphi(s, t) + P \int_{-1}^{1} \varphi(u, t) \cdot \left[\frac{u\varphi(u, s) - t\varphi(t, s)}{t-u}\right]$$

$$+ \varphi(u, s) \left[\frac{s\varphi(s, t) - u\varphi(u, t)}{s-u}\right] du.$$

Using $u/(t-u) = t/(t-u) - 1$ and $u/(s-u) = s/(s-u) - 1$, we get

$$I(s, t) = \varphi(t, s) - \varphi(s, t) + \int_{-1}^{1} s\varphi(u, s) \frac{\varphi(s, t) - \varphi(u, t)}{s-u}$$

$$- t\varphi(u, t) \frac{\varphi(t, s) - \varphi(u, s)}{t-u}\, du.$$

Using the integral equation (7) for $\varphi(u, s)$ we get

$$\frac{\varphi(s, t) - \varphi(u, t)}{s-u} = \frac{k(s, t) - k(u, t)}{s-u} + \int_{-1}^{1} \frac{h(s, u', t) - h(u, u', t)}{s-u} \varphi(u', t)\, du'$$

and a similar expression for the first term in the integral for $I(s, t)$. Since $k(s, t) = k(t, s)$,

$$I(s,t) = \int_{-1}^{1} \varphi(u, s)\, du \int_{-1}^{1} s\, \frac{h(s, u', t) - h(u, u', t)}{s - u}\, \varphi(u', t)\, du'$$
$$- \int_{-1}^{1} \varphi(u, t)\, du \int_{-1}^{1} t\, \frac{h(t, u', s) - h(u, u', s)}{t - u}\, \varphi(u', s)\, du'.$$

Principal values of the integrals involved here need not be taken, since $h(u, u', s)$ is analytic in all three variables. Invert the order of integration in the second integral and interchange the dummy variables of integration u and u'. Then

$$I(s, t) = \int_{-1}^{1} \varphi(u, s)\, du \int_{-1}^{1} \varphi(u', t) \Big[s\, \frac{h(s, u', t) - h(u, u', t)}{s - u}$$
$$- t\, \frac{h(t, u, s) - h(u', u, s)}{t - u'} \Big]\, du'.$$

It is easily shown that the quantity in square brackets vanishes identically in s. This proves the theorem.

REMARK. The singularity of $F(t)$ at $t = 0$ indicated by Theorem 3.3 is only apparent. In fact, we have

$$(uf, \varphi_t) = tC(t) f(t) + tP \int_{-1}^{1} \frac{uf(u)\, \varphi(u, t)}{t - u}\, du;$$

hence

$$F(t) = \frac{C(t) f(t) + P \int_{-1}^{1} \frac{uf(u)\, \varphi(u, t)}{t - u}\, du}{C^2(t) + \pi^2 t^2 \pi^2(t, t)},$$

By Lemma 2.2, $F(t)$ is Holder continuous on $-1 < t < 1$.

We now come to the main result of this paper.

THEOREM 3.4. *Let $f(u)$ be Hölder continuous on $-1 < u < 1$. Then $f(u)$ can be expanded*

$$f(u) = \sum_{i=1}^{n} \sum_{j=1}^{m_1} F_{ij} \varphi_{ij} + \int_{-1}^{1} F(s)\, \varphi_s(u)\, ds,$$

where $s[C^2(s) + \pi^2 s^2 \varphi^2(s, s)]\, F(s) = (uf, \varphi_s)$; and the F_{ij} are uniquely determined.

PROOF. The function $F(s)$ is Hölder continuous on $-1 < s < 1$ by the remark preceding Theorem 3.4. Set

$$f_0^*(u) = f(u) - \int_{-1}^{1} F(s)\, \varphi_s(u)\, ds.$$

By Theorem 3.3, $(uf_0^*, \varphi_s) = 0$ on $-1 < s < 1$; moreover, f_0^* is Hölder continuous. By Theorem 3.2, $f_0^* \in \mathcal{N}_0$. Thus

$$f_0^* = \sum_{i=1}^{n} \sum_{j=1}^{m_i} F_{ij}\varphi_{ij}.$$

The F_{ij} are uniquely determined since the φ_{ij} are linearly independent. Q.E.D.

We shall now discuss methods of computing the coefficients F_{ij}. The F_{ij} can always be found as the solution to the system of equations

$$(f_0^*, \varphi_{ij}) = \sum_{k,\ell} F_{k\ell}(\varphi_{k\ell}, \varphi_{ij}) \qquad \begin{aligned} i &= 1, \cdots, n \\ j &= 1, \cdots, m_i. \end{aligned}$$

The $N \times N$ matrix $\| (\varphi_{k\ell}, \varphi_{ij}) \|$ (the functions φ_{ij} being arranged in some suitable linear order) is nonsingular since the $\{\varphi_{ij}\}$ are linearly independent. Under certain conditions, however, the following orthogonality relations can be used to obtain a triangular matrix for the determination of the F_{ij}.

THEOREM 3.5. *If $s_i \neq s_k$ then $(u\varphi_{ij}, \varphi_{k\ell}) = 0$ for $j = 1, \cdots, m_i$, $k = 1, \cdots, m_k$. Moreover, the $m_i \times m_i$ matrix $\| (u\varphi_{ij}, \varphi_{ik}) \|$, $j, k = 1, \cdots, m_i$, is triangular: $(u\varphi_{ij}, \varphi_{ik}) = 0$ for $j + k < m_i$.*

PROOF. All h.o. eigenfunctions, except possibly those corresponding to $s = \infty$, are in the domain of the operator Ru. Moreover, $Ru\,\bar\varphi_{i1} = \bar s_i \bar\varphi_{i1}$, and $Ru\,\bar\varphi_{ij} = \bar s_i \bar\varphi_{ij} + \bar\varphi_{ij-1}$, where $\bar s_i$ denotes the complex conjugate of s_i. If $s_i, s_k \neq \infty$,

$$\begin{aligned} s_i(u\varphi_{i1}, \varphi_{k1}) &= (u\, Ru\, \varphi_{i1}, \varphi_{k1}) = (u\, \varphi_{i1}, Ru\, \varphi_{k1}) \\ &= (u\varphi_{i1}, \bar s_k \varphi_{k1}) = s_k(u\varphi_{i1}, \varphi_{k1}). \end{aligned}$$

Hence if $s_i = s$ $(u\varphi_{i1}, \varphi_{k1}) = 0$. The remaining orthogonality relations may be established inductively by rows, using the equations (3) for the φ_{ij}. If $s_k = \infty$, we have

$$(u\varphi_{i1}, \varphi_{k1}) = s_i((I - K)\, \varphi_{i1}, \varphi_{k1}) = s_i(\varphi_{i1}, (I - K)\, \varphi_{k1}) = 0.$$

To establish the relations in the second statement, write

$$s_i(u\varphi_{i1}, \varphi_{ij}) = (u\, Ru\, \varphi_{i1}, \varphi_{ij}) = (u\varphi_{i1}, \bar{s}_i\varphi_{ij} + \varphi_{ij-1})$$
$$= s_i(u\varphi_{i1}, \varphi_{ij}) + (u\varphi_{i1}, \varphi_{ij-1});$$

that is $(u\varphi_{i1}, \varphi_{ij}) = 0$ for $j = 1, 2, \cdots, m_i - 1$. Now suppose that $(u\varphi_{ij}, \varphi_{ik}) = 0$. Then using equations (3) we can show

$$s_i(u\varphi_{ij}, \varphi_{ik+1}) = s_i(u\varphi_{ij}, \varphi_{ik+1}) - (u\varphi_{ij-1}, \varphi_{ik+1});$$

from which $(u\varphi_{ij-1}, \varphi_{ik+1}) = 0$. Hence by induction along the anti-diagonals we can derive the results in the second part of Theorem 3.6. The same results also hold for h.o. eigenfunctions at ∞; the proof is left to the reader.

Note that the functions $u\phi_{ij}$ are h.o. eigenfunctions for the adjoint operator $(Ru)^* = uR$. Theorem 3.6 thus states orthogonality properties between eigenfunctions of an operator T and its adjoint T^*.

If for fixed i the matrix $\| (u\varphi_{ij}, \varphi_{ik}) \|$ is nonsingular, then the F_{ij} can be determined from the system of equations

$$(uf_0^*, \varphi_{ij}) = \sum_k F_{ik}(u\varphi_{ik}, \varphi_{ij}), \qquad j = 1, \cdots, m_i .$$

The triangular matrix $\| (u\varphi_{ij}, \varphi_{ik}) \|$ is, of course, somewhat easier to invert. The remaining F_{ij} will have to be computed by the first method above.

IV. Solutions of the Transport Equation

If s_i is an eigenvalue of (1) it is easily seen that $f(x, u) = e^{-x/s_i}\varphi_{i1}(u)$ is a solution of (2) if $s_i \neq \infty$, and that $f(x, u) = \varphi_{i1}(u)$ is a solution of (2) if $s_i = \infty$. Solutions of (2) corresponding to h.o. eigenfunctions can be constructed using the residue method discussed in the following theorem.

Theorem 4.1. *Let s_i be an eigenvalue of (1), $j = 1, \cdots, m_i$; let*

$$f_i(u) = \sum_j F_{ij}\varphi_{ij}(u).$$

Let C_i be a simple closed contour enclosing the eigenvalue s_i. Then

$$f(x, u) = \frac{1}{2\pi i}\int_{C_i} e^{-x/s}(s - u - sK)^{-1}(I - K) f_i\, ds$$

is a solution of (2) which satisfies $f(0, u) = f_i(u)$.

PROOF. The resolvent operator $(s - u - sK)^{-1}$ is analytic in s in the complex plane cut at $[-1, 1]$; it has poles at the eigenvalues $s = s_i$.
This can be established using standard techniques of linear integral equations.

Formal substitution of $f(x, u)$ into the transport equation gives as the result

$$u \frac{\partial}{\partial x} f(x, u) + f(x, u) - Kf(x, u)$$

$$= \frac{1}{2\pi i} \int_{C_i} e^{-x/s} \frac{1}{s} (s - u - sK)(s - u - sK)^{-1} (I - K) f_i \, ds$$

$$= \frac{1}{2\pi i} \int_{C_i} \frac{e^{-x/s}}{s} (I - K) f_i \, ds = 0. \tag{10}$$

Differentiation under the integral sign is justified, since the integrand

$$-\frac{1}{s} e^{-x/s}(s - u - sK)^{-1} (I - K) f_i = \frac{\partial}{\partial x} \{e^{-x/s}(s - u - sK)^{-1} (I - K) f_i\}$$

is analytic in s. The interchange of the integral operator K with the contour integral is justified, since the integrals involved are absolutely convergent. The second integral in (10) vanishes, since it is analytic in s (C_i does not enclose the origin).

It remains to show that $f(0, u) = f_i(u)$. It is easily shown (for example, by induction) that

$$(s - u - sK)^{-1} (I - K) \varphi_{ij} = \frac{1}{s - s_i} \varphi_{ij} - \frac{1}{(s - s_i)^2} \varphi_{ij-1} + \cdots$$

$$+ \frac{(-1)^{j+1}}{(s - s_i)^j} \varphi_{i1}, \qquad j = 1, 2, \cdots.$$

REMARK. This relatively simple result was made possible by our choice of equations (3) for h.o. eigenfunctions. The system (9) satisfied by the f_{ij} leads to a much more complicated expression for the singularities of the resolvent $(s - u - sK)^{-1} (I - K) = 1/t(I - tu - K)^{-1} (I - K)$ (where $s = 1/t$) at $s = s_i$.

We now have

$$f(0, u) = \frac{1}{2\pi i} \int_{C_i} \sum_j F_{ij}(s - u - sK)^{-1} (I - K) \varphi_{ij} \, ds$$

$$= \sum_j F_{ij} \varphi_{ij}(u) = f_i(u). \qquad \text{Q.E.D.}$$

In case $m_i = 2$, we get for a solution of (2)

$$f(x, u) = (F_{i1} - xF_{i2})\, \varphi_{i1}(u)\, e^{-x/s_i} + F_{i2}\varphi_{i2}(u)\, e^{-s/x_i}$$

As for solutions of the transport equation corresponding to the continuous spectrum, we see that

$$f(x, u) = \int_0^1 e^{-x/s} F(s)\, \varphi_s(u)\, ds$$

will be a solution of (2) in the region $x \geqslant 0$, $-1 < u < 1$, provided that differentiation under a singular integral is justified. However, if $F(s)$ is Hölder continuous, we have

$$\frac{\partial}{\partial x} P \int_0^1 \frac{s\, e^{-x/s} F(s)\, \varphi(u, s)}{s - u}\, ds$$

$$= \frac{\partial}{\partial x} \left\{ \int_0^1 \frac{s\, e^{-x/s} - u\, e^{-x/s}}{s - u} F(s)\, \varphi(u, s)\, ds + u\, e^{-x/u}\, P \int_0^1 \frac{F(s)\, \varphi(u, s)}{s - u}\, ds \right\}$$

$$= P \int_0^1 \frac{e^{-x/s} F(s)\, \varphi(u, s)}{s - u}\, ds.$$

The result also holds when $u = 1$ provided $F(s)$ vanishes sufficiently fast as $s \to 1$; for then the principal value is not necessary when $u = 1$.

V. Applications

Let us consider some applications of the expansion theorem to the solution of some boundary value problems in transport theory. Consider first the problem of finding a solution $f(x, u)$ to the transport equation in the region $x \geqslant 0$ subject to the condition $f(0, u) = f_0(u)$, $0 \leqslant u \leqslant 1$; $f_0(u)$ is given. Physically this corresponds to finding the flux distribution in an infinite homogeneous slab in $x \geqslant 0$ due to an incident flux $f_0(u)$ falling on the interface $x = 0$ from $x = -\infty$. This problem leads to the singular integral equation

$$f_0(u) = \sum_{i,j} F_{ij}\varphi_{ij}(u) + \int_0^1 F(s)\, \varphi_s(u)\, ds, \qquad 0 \leqslant u < 1,$$

for the function $F(s)$ and the coefficients F_{ij}.

By solving this singular integral equation we shall be able to determine the reflected flux distribution given the incident flux distribution. Thus the singular integrals appear to play a dominant role in in the solution of boundary value problems.

In the case of slabs of finite thickness, say $0 \leqslant x \leqslant 1$, we are led to solutions of the form

$$f(x, u) = \int_0^1 e^{-x/s} F_1(s)\, \varphi_s(u)\, ds + \int_{-1}^0 \exp\left(\frac{1-x}{s}\right) F_2(s)\, \varphi_s(u)\, ds$$

Two problems are of special interest:

I. The "criticality" problem, where

$$\begin{aligned} f(0, u) &= 0, & 0 \leqslant u \leqslant 1 \\ f(1, u) &= 0, & -1 \leqslant u \leqslant 0. \end{aligned}$$

II. The "shielding" problem, where

$$\begin{aligned} f(0, u) &= f_0(u) \quad \text{is prescribed for} & 0 \leqslant u \leqslant 1 \\ f(1, u) &= 0, & -1 \leqslant u \leqslant 0. \end{aligned}$$

In problem I, no flux is incident on either interface from the exterior regions. In II, the incident flux at the interface $x = 0$ is prescribed but no flux falls on the interface $x = 1$ from $+\infty$; it is then required to find the transmitted flux $f(1, u)$, $0 \leqslant u \leqslant 1$, and the reflected flux $f(0, u)$, $-1 \leqslant u \leqslant 0$.

Problems I and II naturally lead to the system of singular integral equations

$$f_0(u) = \int_0^1 F_1(s)\, \varphi_s(u)\, ds + \int_{-1}^0 F_2(s)\, e^{1/s}\, \varphi_s(u)\, ds, \qquad 0 \leqslant u \leqslant 1,$$

$$g_0(u) = \int_0^1 e^{-1/s} F_1(s)\, \varphi_s(u)\, ds + \int^0 F_2(s)\, \varphi_s(u)\, ds, \qquad -1 - u \leqslant 0, \tag{11}$$

for the functions $F_2(s)$ given $f_0(u)$ and $g_0(u)$. A sufficient condition for criticality is the existence of a nontrivial solution to (13) when $f_0(u) = 0$ and $g_0(u) = 0$.

If the scattering kernel $k(u, u')$ is simple enough it may be feasible to solve the singular integral equations discussed above. For example, if

$$k(u, u') = \frac{\lambda_0}{2} + \lambda_1 \frac{3}{2}\, uu'$$

then the methods discussed in part III lead to

$$\varphi(u, s) = \frac{\lambda_0}{2} + \frac{3}{2} \lambda_1 (1 - \lambda_0)\, us$$

$$C(s) = 1 + 3s^2 \lambda_1 (1 - \lambda_0) + \frac{1}{2} (\lambda_0 + 3\lambda_i (1 - \lambda_0)\, s^2)\, s \log \frac{1 - s}{1 + s}.$$

The corresponding integral equations in this case are fairly simple. We hope to be able to discuss these matters in a future paper.

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